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Solitons of the reduced Maxwell–Bloch equations for circularly polarized light

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Abstract

It is shown that the reduced Maxwell–Bloch equations for circularly polarized light are integrable by the inverse scattering transform method. A Bäcklund transformation is given and used to construct a hierarchy of N-soliton solutions. The breather solution corresponding to the 2π-pulse of self-induced transparency is discussed in some detail.

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1. Introduction

The phenomenon of self-induced transparency [1, 2] has been successfully described by a slowly varying envelope approximation (SVEA) of the Maxwell–Bloch equations. The so-called SIT equations resulting in this way are integrable by the inverse scattering method [3, 4] even taking into account inhomogeneous broadening of the atomic resonance frequencies. Eilbeck *et al* [5] proposed a more accurate approximation called reduced Maxwell–Bloch (RMB) equations. They avoided SVEA and—as a weaker assumption—neglected a backscattered wave. For plane polarized waves it has been found that the RMB equations are integrable as well and, just as the SIT equations, are connected with a Zakharov–Shabat scattering problem.

On the basis of the SIT equations there is—from the mathematical point of view—no difference in principle between the treatment of a circularly polarized wave and a plane polarized one interacting with the appropriate atomic systems. The same does not hold for the RMB equations, and one can find in the literature the statement that "the rotating RMB equations . . . do not have soliton solutions and are not an integrable system" [6]. In this paper it is demonstrated that the rotating RMB equations—in the sharp-line limit, at least—are an integrable system and do have soliton solutions. They are not, however, connected with a Zakharov–Shabat but with a Kaup–Newell scattering problem [7].

For the whole subject of Darboux and Bäcklund transformations together with their differential-geometric background, their origins in the 19th and early 20th centuries and a rich field of applications, we refer the reader to the book by Rogers and Schief [8]. Crum [9] expressed an N-fold Darboux transformation for the Sturm–Liouville problem (or *Schrödinger equation*) in terms of Wronskian determinants. Later on similar procedures were developed for the scattering problems of Zakharov–Shabat and more general types [10–14]. Here Vandermonde-like determinants appear instead of Wronskians.

In section 2 we rederive the rotating RMB equations and set them in an appropriate form. In section 3 we establish a simultaneous system of linear differential equations such that the rotating RMB equations appear as the integrability conditions and where the τ-part takes the form of a Kaup–Newell scattering problem. The construction of an N-fold Bäcklund transform in section 4 parallels the procedures developed in preceding papers [15–17]. The connecting link is the Kaup–Newell problem. The simplest solutions of the resulting hierarchy are discussed in section 5, where, in particular, we see that the breather solution corresponds to the 2π-pulse of self-induced transparency just as this has been long known for the RMB equations with plane polarization [5]. The notation of Vandermonde-like determinants as used in this paper is explained in the appendix.

2. Maxwell–Bloch and reduced Maxwell–Bloch (RMB) equations

We consider an electromagnetic wave propagating in the z-direction and interacting with a nondegenerate two-level system. To dissolve degeneracy a static magnetic field is applied in the z-direction. For definiteness we may think of a dipole transition $(J = 1/2, M = -1/2) \leftrightarrow (J = 1/2, M = +1/2)$, where J is the total angular momentum, and M is its z-component. In this paper we will not take into account inhomogeneous broadening, so that there is a common resonance frequency ω_0 for all atoms. The Maxwell–Bloch equations for this system may be written as follows:

$$\begin{aligned} \left(\partial_z^2 - \frac{1}{c^2}\partial_t^2\right)\mathcal{E}_x &= \frac{4\pi dn}{c^2}\partial_t^2 R_x \\ \left(\partial_z^2 - \frac{1}{c^2}\partial_t^2\right)\mathcal{E}_y &= \frac{4\pi dn}{c^2}\partial_t^2 R_y \end{aligned} \tag{1}$$

$$\begin{aligned} \partial_t R_x &= -\omega_0 R_y - \frac{2d}{\hbar}\mathcal{E}_y R_z \\ \partial_t R_y &= \omega_0 R_y + \frac{2d}{\hbar}\mathcal{E}_x R_z \\ \partial_t R_z &= \frac{2d}{\hbar}(R_x\mathcal{E}_y - R_y\mathcal{E}_x). \end{aligned} \tag{2}$$

Here $\mathcal{E}_x$ and $\mathcal{E}_y$ are the electric field components, and (R_x, R_y, R_z) is the Bloch vector. c is the velocity of light in vacuum or, more generally, in the host medium, and d is the dipole moment. The second-order differential operators on the left-hand sides of (1) may be factorized, and as the crucial approximation [5] the factor $(\partial_x - (1/c)\partial_t)$ is replaced by $-(2/c)\partial_t$. Afterwards one may integrate once to get

$$(c\partial_z + \partial_t)\mathcal{E}_x = -2\pi dn\partial_t R_x \qquad (c\partial_z + \partial_t)\mathcal{E}_y = -2\pi dn\partial_t R_y. \tag{3}$$

The combined equations (2), (3) then are the RMB equations for a circularly polarized wave. From them—in a formal sense only—one may get the RMB equations for a plane polarized wave by ignoring the second of equations (3) and putting $\mathcal{E}_y = 0$ in (2). The essential difference is that in the latter case one may get rid of the time derivative of R_x by use of $\partial_t R_x = -\omega_0 R_y$, while for circular polarization such a possibility does not exist.

We introduce characteristic coordinates χ, τ together with a proper scaling,

$$\chi = (4\pi d^2 n/\hbar c)z \qquad \tau = \omega_0(t - z/c) \tag{4}$$

$$E_{x,y} = (2d/\hbar\omega_0)\mathcal{E}_{x,y} \tag{5}$$

and we combine the x- and y-components by a complex notation,

$$E = E_x + \mathrm{i}E_y \qquad R = R_x + \mathrm{i}R_y. \tag{6}$$

Finally we write down a more general system of partial differential equations for five functions (E, F, R, S, R_z),

$$\begin{aligned}
&\partial_\tau R = \mathrm{i}(R + ER_z)\\
&\partial_\tau S = -\mathrm{i}(S + FR_z)\\
&\partial_\tau R_z = \tfrac{\mathrm{i}}{2}(RF - SE)\\
&\partial_\chi E = -\partial_\tau R\\
&\partial_\chi F = -\partial_\tau S
\end{aligned} \tag{7}$$

such that together with the reduction $F = E^*$, $S = R^*$, R_z real it becomes equivalent to (2), (3). The asterisk denotes complex conjugation.

3. The linear system, Riccati equations and Darboux/Bäcklund transformations

We are starting from the system (7) where (E, F, R, S, R_z) are considered as independent functions of χ and τ with no reduction so far. Obviously, there are two conservation laws,

$$\partial_\tau\big(RS + R_z^2\big) = 0 \tag{8}$$

$$\partial_\chi(EF) + 2\partial_\tau R_z = 0. \tag{9}$$

Equations (7) are the integrability conditions for the simultaneous linear partial differential equations

$$\partial_\tau\phi = U\phi \qquad U \equiv \frac{1}{2}\begin{pmatrix} -\mathrm{i}\zeta^2 & E\zeta \\ F\zeta & \mathrm{i}\zeta^2 \end{pmatrix} \tag{10}$$

$$\partial_\chi\phi = V\phi \qquad V = \frac{-\zeta W}{2(1+\zeta^2)} \tag{11}$$

$$W = \begin{pmatrix} \mathrm{i}\zeta R_z & R \\ S & -\mathrm{i}\zeta R_z \end{pmatrix} \tag{12}$$

where $\phi = (\varphi_1, \varphi_2)^T$ denotes a two-component column vector. Equation (10) has the form of a Kaup–Newell scattering problem [7]. From (10)–(12) one may easily derive a system of Riccati equations for the quotient $\beta(\chi, \tau) = \varphi_2/\varphi_1$,

$$\partial_\tau\beta = \mathrm{i}\zeta^2\beta + \zeta(-E\beta^2 + F)/2 \tag{13}$$

$$\partial_\chi\beta = \frac{\zeta}{(1+\zeta^2)}[\mathrm{i}\zeta R_z\beta + (R\beta^2 - S)/2]. \tag{14}$$

Then the RMB equations (7) as well are integrability conditions for the simultaneous Riccati equations (13), (14).

Reduction. If $F = E^*$, $S = R^*$, R_z real and ζ, β solves (13), (14) then ζ^*, $1/\beta^*$ is a solution as well.

In order to establish a Bäcklund transformation we assume that for a given solution $\{E, F, R, S, R_z\}$ to (7) one particular solution $\{\beta_1(\chi, \tau), \zeta_1\}$ to (13), (14) is known, and we define the matrix

$$M = M(\zeta) = \begin{pmatrix} \zeta\beta_1 & -\zeta_1 \\ -\zeta_1 & \zeta\alpha_1 \end{pmatrix} \qquad \alpha_1 \equiv 1/\beta_1. \tag{15}$$

Note that in (10), (13) the variable χ does play the role of a parameter and, vice versa, in (11), (14) τ is a parameter only. Let us, for a moment, ignore these parameters and formulate two theorems giving Darboux transformations for the *scattering problems* (10) and (11), respectively.

Theorem 1 [15]. *From any solution* $\{\phi(\tau), \zeta, E(\tau), F(\tau)\}$ *to* (10) *a new solution* $\{\phi^{[1]}(\tau), \zeta, E^{[1]}(\tau), F^{[1]}(\tau)\}$ *is found by the Darboux transformation*

$$\begin{aligned} \phi^{[1]} &= M\phi \\ E^{[1]} &= \beta_1(\beta_1 E - 2\mathrm{i}\zeta_1) \\ F^{[1]} &= \alpha_1(\alpha_1 F + 2\mathrm{i}\zeta_1). \end{aligned} \tag{16}$$

Theorem 2. *From any solution* $\{\phi(\chi), \zeta, R(\chi), S(\chi), R_z(\chi)\}$ *to* (11) *a new solution* $\{\phi^{[1]}(\chi), \zeta, R^{[1]}(\chi), S^{[1]}(\chi), R_3^{[1]}(\chi)\}$ *is found by the Darboux transformation*

$$\begin{aligned} \phi^{[1]} &= M\phi \\ R^{[1]} &= \left(\zeta_1^2 + 1\right)^{-1}\left(\beta_1^2 R + 2\mathrm{i}\zeta_1\beta_1 R_z + \zeta_1^2 S\right) \\ S^{[1]} &= \left(\zeta_1^2 + 1\right)^{-1}\left(\alpha_1^2 S - 2\mathrm{i}\zeta_1\alpha_1 R_z + \zeta_1^2 R\right) \\ R_z^{[1]} &= \left(\zeta_1^2 + 1\right)^{-1}\left(\mathrm{i}\zeta_1(R\beta_1 - S\alpha_1) + \left(1 - \zeta_1^2\right)R_z\right). \end{aligned} \tag{17}$$

Both these theorems may be proved by direct verification. When—as above it has been originally assumed—$\beta_1(\chi, \tau)$ solves both (13) and (14) with ζ replaced by ζ_1 these same formulae (16), (17) together define a Bäcklund transformation. It is easily seen that the reduction $F = E^*, S = R^*, R_z$ real is conserved under the Darboux/Bäcklund transformations.

It has been found [15] and is not difficult to check that Darboux transformations in the sense of theorem 1 commute.

4. The *N*-fold Darboux/Bäcklund transform

Now we return to the spectral problem (10) (with no reduction so far). If $\phi_1 \equiv (\varphi_{11}, \varphi_{21})^T$ solves (10) with $\zeta = \zeta_1$, then $\beta_1 = \varphi_{21}/\varphi_{11}$ solves (13) with ζ replaced by ζ_1, and for the matrix defined by (15) it holds $M(\zeta_1)\phi_1 = 0$. Now we assume that N solutions $\{\phi_j, E, F, \zeta_j\}$, $j = 1, \ldots, N$, to (10) are known. The wavefunction of the N-fold Darboux transform is an Nth-order polynomial in ζ,

$$\phi^{[N]} = M^{[N]}(\zeta)\phi \qquad M^{[N]}(\zeta) \equiv \sum_{k=0}^{N} P_k \zeta^{N-k}. \tag{18}$$

From $M(\zeta_1)\phi_1 = 0$ together with commutativity, it follows that

$$M^{[N]}(\zeta_j)\phi_j = 0. \tag{19}$$

From the iteration of (15), (16) the coefficients P_k get the structure

$$P_{2l-1} = \begin{pmatrix} 0 & p_{2l-1} \\ s_{2l-1} & 0 \end{pmatrix} \qquad P_{2l} = \begin{pmatrix} p_{2l} & 0 \\ 0 & s_{2l} \end{pmatrix} \qquad p_N = s_N = \text{const}. \tag{20}$$

Equation (19) decomposes into two separate systems of linear equations for the coefficients p_k and s_k, respectively, and these two systems may be solved according to Cramer's rule. We have to distinguish whether N is odd or even. In order to conserve space we will treat the case of even N only and put $N = 2n$, $p_{2n} = s_{2n} = -1$,

$$\sum_{l=0}^{n} p_{2l}\zeta_j^{2(n-l)} + \sum_{l=1}^{n} p_{2l-1}\zeta_j^{2(n-l)+1}\beta_j = 0 \tag{21}$$

$$\sum_{l=0}^{n} s_{2l}\zeta_j^{2(n-l)} + \sum_{l=1}^{n} s_{2l-1}\zeta_j^{2(n-l)+1)}\alpha_j = 0. \tag{22}$$

For later use we will write down the coefficients p_0 and p_1 explicitly, and we will use the notation of Vandermonde-like determinants (see the appendix),

$$p_0 = (-1)^n \frac{\mathcal{V}_{nn}\left(1\ldots1;\, \zeta_j\beta_j \middle| \zeta_j^2\right)}{\mathcal{V}_{nn}\left(\zeta_j^2;\, \zeta_j\beta_j \middle| \zeta_j^2\right)} \tag{23}$$

$$p_1 = \frac{\mathcal{V}_{n+1,n-1}\left(1\ldots1;\, \zeta_j\beta_j \middle| \zeta_j^2\right)}{\mathcal{V}_{nn}\left(\zeta_j^2;\, \zeta_j\beta_j \middle| \zeta_j^2\right)}. \tag{24}$$

To obtain s_0 and s_1, one only needs to replace the letter β by α in (23), (24). More generally it holds

$$s_k = p_k(\alpha \longrightarrow \beta). \tag{25}$$

Let us write the transformed spectral problem in the form

$$\partial_\tau \phi_\tau^{[N]} = U^{[N]}\phi^{[N]} \equiv (J\zeta^2 + Q^{[N]}\zeta)\phi^{[N]} \tag{26}$$

using the abbreviations

$$J \equiv \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} \qquad Q^{[N]} \equiv \begin{pmatrix} 0 & E^{[N]} \\ F^{[N]} & 0 \end{pmatrix}. \tag{27}$$

Substitution of (18) into (10), (26) and comparison of powers in ζ lead to

$$\begin{aligned}
&[P_0, J] = 0 \\
&[P_1, J] + P_0 Q - Q^{[N]} P_0 = 0 \\
&\partial_\tau P_{k-1} + [P_{k+1}, J] + P_k Q - Q^{[N]} P_k = 0 \qquad k = 1, \ldots, N-1 \\
&\partial_\tau P_{N-1} + P_N Q - Q^{[N]} P_N = 0.
\end{aligned} \tag{28}$$

and from the second equation of this system we get

$$E^{[N]} = \frac{p_0 E + 2\mathrm{i} p_1}{s_0} \qquad F^{[N]} = \frac{s_0 F - 2\mathrm{i} s_1}{p_0}. \tag{29}$$

The above formulae determine the N-fold Darboux transformation. When we 'switch on' the time t we know from section 3 that each of the N single-transformation steps becomes a Bäcklund transformation, i.e. it transforms the simultaneous system (10), (11) with preserving its form. Consequently, our result gives the N-fold Bäcklund transformation as well. Explicit formulae are available for the matrix elements of $M^{[N]}$ as defined by (18), (20). For this purpose we only have to read, e.g., (21) together with $M_{11}^{[N]} = \sum p_{2l}\zeta^{2(n-l)}$ as a linear system of equations for $p_1, \ldots, p_{2n-1}, M_{11}^{[N]}$ with $p_{2n} = -1$ and to recognize that the solution again can be expressed in terms of Vandermonde-like determinants. In this way we find

$$M_{11}^{[N]}(\zeta) = -\frac{\mathcal{V}_{n+1,n}\left(1\ldots1;\, \beta_j\zeta_j, 0 \middle| \zeta_j^2, \zeta^2\right)}{\mathcal{V}_{n,n}\left(\zeta_j^2;\, \beta_j\zeta_j \middle| \zeta_j^2\right)} \tag{30}$$

and, quite analogously,

$$M_{12}^{[N]}(\zeta) = -\frac{\mathcal{V}_{n+1,n}\left(1\ldots 1, 0; \beta_j\zeta_j, \zeta \middle| \zeta_j^2, \zeta^2\right)}{\mathcal{V}_{n,n}\left(\zeta_j^2; \beta_j\zeta_j \middle| \zeta_j^2\right)} \tag{31}$$

$$M_{21}^{[N]} = M_{12}^{[N]}(\beta \to \alpha) \qquad M_{22}^{[N]} = M_{11}^{[N]}(\beta \to \alpha). \tag{32}$$

The transformation law of the matrix function W is found as

$$W^{[N]} = \left(-2(\zeta + 1/\zeta)M_\chi^{[N]} + M^{[N]}W\right)[M^{[N]}]^{-1} \tag{33}$$

or, if we fix $\zeta = \mathrm{i}$,

$$W^{[N]}(\mathrm{i}) = M^{[N]}(\mathrm{i})W(\mathrm{i})[M^{[N]}(\mathrm{i})]^{-1}. \tag{34}$$

From this last equation we easily obtain the transformation law of the atomic state,

$$R_z^{[N]} = ((M_{11}M_{22} + M_{12}M_{12})R_z + M_{11}M_{21}R - M_{12}M_{22}S)/D \tag{35}$$

$$R^{[N]} = \left(M_{11}^2 R - M_{12}^2 S + 2M_{11}M_{12}R_z\right)\big/D \tag{36}$$

$$S^{[N]} = \left(-M_{21}^2 R + M_{22}^2 S - 2M_{21}M_{22}R_z\right)\big/D \tag{37}$$

$$D \equiv M_{11}M_{22} - M_{12}M_{21} \qquad M_{ik} \equiv M_{ik}^{[N]}(\mathrm{i}). \tag{38}$$

If now we choose the reduction $F = E^*$, $S = R^*$, R_z real, we have to take the eigenvalues as real or as pairs of complex conjugate values and to choose

(i) $|\beta_j| = 1$ for real ζ_j or
(ii) $\beta_l = 1/\beta_k^* = \alpha_k^*$ when $\zeta_l = \zeta_k^*$.

Then we get $s_j = p_j^*$. Consequently, the required symmetry is conserved.

5. Examples

5.1. Harmonic waves ($N = 1$)

In three foregoing papers [15–17] referring to the Kaup–Newell scattering problem, it has been found that the application of a one-step Bäcklund transformation on the vacuum state does not yield a soliton but a harmonic wave, and this happens here as well. From (16), (17) with the vacuum $E = F = R = S = 0$, $R_z = -1$ as the seed solution and with real ζ_1, we get the wave

$$\begin{aligned} &E^{[1]} = -2\mathrm{i}\zeta_1\beta_1 && F^{[1]} = E^{[1]*} \\ &R^{[1]} = E^{[1]}\big/\left(\zeta_1^2 + 1\right) && S^{[1]} = R^{[1]*} \\ &R_z^{[1]} = \left(\zeta_1^2 - 1\right)\big/\left(\zeta_1^2 + 1\right) \end{aligned} \tag{39}$$

$$\beta_1 \equiv \exp\left[\mathrm{i}\zeta_1^2\left(\tau - \chi\big/\left(1 + \zeta_1^2\right)\right)\right]. \tag{40}$$

It has been observed already by Bullough *et al* [6] that for the rotating RMB equations there is such a harmonic wave instead of a sech solitary wave solution.

5.2. The breather ($N = 2$)

Again we start from the vacuum. Now we choose $N = 2$, $\zeta_1 = \mathrm{i}\rho \exp(\mathrm{i}\varphi)$, $\zeta_2 = \zeta_1^*$, $\beta_2 = 1/\beta_1^*$ with β_1 given by (40). We define real linear functions $\theta_i(\chi, \tau)$, $\theta_r(\chi, \tau)$ by

$$\theta_i - \mathrm{i}\theta_r \equiv \zeta_1^2\big(\tau - \chi\big/\big(1 + \zeta_1^2\big)\big). \tag{41}$$

or

$$\theta_r = \tau \sin 2\varphi - (\chi/2)\cot\varphi \tag{42}$$

$$\theta_i = -\tau \cos 2\varphi - \chi/2. \tag{43}$$

Then it holds

$$\beta_1 = \exp(\mathrm{i}\theta_i + \theta_r), \tag{44}$$

and from (29), (35), (36) we get

$$E^{[2]} = -2\mathrm{i}\rho \sin(2\varphi)\, \mathrm{e}^{-\mathrm{i}\theta_i} \frac{\cosh(\theta_r + \mathrm{i}\varphi)}{\cosh^2(\theta_r - \mathrm{i}\varphi)} \tag{45}$$

$$R_z^{[2]} = \left[\frac{\cos^2\varphi - \sinh^2\theta_r}{\cos^2\varphi + \sinh^2\theta_r} - \frac{1}{4}(\rho - 1/\rho)^2\right] \Bigg/ \left[1 + \left(\frac{\rho - 1/\rho}{2\sin\varphi}\right)^2\right] \tag{46}$$

$$R^{[2]} = 2\sin(2\varphi)\, \mathrm{e}^{-\mathrm{i}\theta_i} \frac{\cosh(\theta_r - \mathrm{i}\varphi) - \rho^{-2}\cosh(\theta_r + \mathrm{i}\varphi)}{\cosh^2(\theta_r - \mathrm{i}\varphi)[(\rho - 1/\rho)^2 + 4\sin^2(\varphi)]}. \tag{47}$$

Equations (41)–(47) define the general complete breather solution. From (45) we find a rather simple formula for the pulse shape,

$$|E^{[2]}|^2 = \frac{4\rho^2 \sin^2(2\varphi)}{\sinh^2(\theta_r) + \cos^2(\varphi)}. \tag{48}$$

From (46) we see that the maximum value of $R_z^{[2]}$ is achieved at $\theta_r = 0$ and that it is equal to 1 for $\rho = 1$ but is less than 1 otherwise. From the physical point of view, according to our scaling (5), it is realistic to assume $|E^{[2]}| \ll 1$, i.e. $|\varphi| \ll 1$. Now we will consider the breather under these specifications, $\rho = 1$, $|\varphi| \ll 1$, and take series expansions with respect to φ,

$$\theta_i = \chi/2 - \tau(1 - \varphi^2) + O(\varphi^3) \tag{49}$$

$$\theta_r = -2\varphi\tau - (\chi/2\varphi)(1 - \varphi^3/3)) + O(\varphi^3) \tag{50}$$

$$E^{[2]} = 4\mathrm{i}\varphi\, \mathrm{e}^{-\mathrm{i}\theta_i} \operatorname{sech}\theta_r (1 + 3\mathrm{i}\varphi \tanh\theta_r) + O(\varphi^3) \tag{51}$$

$$R_z^{[2]} = -1 + 2\operatorname{sech}^2\theta_r (1 - \varphi^2 \tanh^2\theta_r) + O(\varphi^3) \tag{52}$$

$$R^{[2]} = 2\mathrm{e}^{-\mathrm{i}\theta_i} \operatorname{sech}\theta_r \tanh\theta_r (1 + 2\mathrm{i}\varphi \tanh\theta_r) + O(\varphi^2). \tag{53}$$

The solution described by (49)–(53) is the usual 2π-pulse of SIT with slight next-order corrections. In particular, we see that (51) contains a weak chirp. In a sense our result once more proves that the SIT equations are a rather good approximation.

6. Summary and conclusions

We have shown that the rotating RMB equations are integrable by the inverse spectral transform method and that this is another application of the Kaup–Newell scattering problem. In a paper in preparation [18] it will be shown that, more generally, an anisotropy of the polarizability can be included without destroying integrability. In this paper we used Bäcklund transformations to establish N-soliton formulae in terms of Vandermonde-like determinants. It has been demonstrated in recent papers [15–17] that such formulae are well suited for numerical evaluation and for generating computer pictures up to $N = 8$, at least. Clearly, this could be done in the present case as well. Instead, here we discussed only the breather solution ($N = 2$) which corresponds to the 2π-pulse of the SIT equations. So far, qualitatively the situation is the same as is known for the RMB equations with plane polarization. There is, however, an important difference with respect to inhomogeneous broadening which is easily taken into account for plane polarization but, probably, cannot be included for the rotating RMB equations without losing integrability.

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Appendix. Vandermonde-like determinants

Vandermonde-like determinants are defined as follows [14],

$$\mathcal{V}_{MN}(a_r; b_r|x_r) := \begin{vmatrix} a_1 & a_1x_1 & \cdots & a_1x_1^{M-1} & b_1 & b_1x_1 & \cdots & b_1x_1^{N-1} \\ a_2 & a_2x_2 & \cdots & a_2x_2^{M-1} & b_2 & b_2x_2 & \cdots & b_2x_2^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{M+N} & a_{M+N}x_{M+N} & \cdots & a_{M+N}x_{M+N}^{M-1} & b_{M+N} & b_{M+N}x_{M+N} & \cdots & b_{M+N}x_{M+N}^{N-1} \end{vmatrix} \tag{A1}$$

where $r = 1, 2, \ldots, M+N$. These determinants have several remarkable structural properties listed in [14]. In particular, any Vandermonde-like determinant $\mathcal{V}_{MN}$ can be expressed as a sum over binary products of genuine Vandermonde determinants $\mathcal{V}_N$.

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